Notes on Projective Geometry

Towards computing the fundamental group of the real projective plane

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Projective geometry is a branch of mathematics that study the properties of geometric objects that remain the same under diffferent perspectives (such properties are invariant with respect to projective transformation).

The subject formalizes ideas from perspective art, such as implicit horizon points or lines "at infinity" that induce perspective. In a perspective drawing, two parallel lines meet at a distant horizon point that is not in the picture. (These points and lines at infinity that guide perspective drawings are called "idealized directions" and "idealized points" respectively.)

The study of projective geometry dates back to the Greek mathematician Pappus of Alexandria in the 3rd century. There was a revival of the subject amongst French mathematicians, beginning with Desargues in the early 1600s and continuing with Poncelet and Carnot (where Carnot was balancing a foundational exposition on the subject with leadership position in Robespierre's Committee of Public Safety. One would be suprised but cannot expect an less from a French mathematician).

As the projective literature grew, it began to make contributions to new fields of analytical and algebraic geometry, as well as gain popularity amongst mathematical physicists in the development of quantum mechanics. In fact, Dirac would often draw projective diagrams of his equations to develop intuition before formalizing them algebraically and would leave this diagrams out of print because they were expensive and he assumed many physicists were unfamiliar with the subject. Klein's Erlangen program also developed projective geometry, in addition to Euclidean and affine geometry, around this time (1872).

We are interested in deriving the basic axioms and properties of generalized projective spaces, initially following notes from Hitchin, towards computing the fundamental group of the real projective plane. We will discover beautiful results that touch quantum physics and traditional Binasuan dancing in addition to algebraic topology.

1 Basic Definitions

Definition 1.1. Let V be a vector space. The projective space $P(V)$ is the set of 1-dimensional vector subspaces of V.

It can be useful to think of projective spaces, at least in the real case, as bundles of lines that pass through the origin. This reduces our space by a dimension, motivating the common shorthand for the reals $P^n(R) = P(R^{n+1})$.

Many sources encourage thinking about the sphere S^n for a projective space $P^n(\mathbb{R})$ to ease the visualization of a space of lines. Antipodal points on the sphere are identified and every such antipodal pair is injective with the actual elements of the projective space.

I have found it easier to just think of the vector subspaces the projective elements represent (projective points and lines are lines and planes through the origin in $\mathbb{R}^3$) for more natural geometric intuition.

1.1 Decomposition

The following decomposition is useful to understand the structure of theses spaces:

$$P(R^n) = R^{n-1} + P(R^{n-1})$$

Essentially our goal is to take R^n and partition the set of points into 1-dim vector subspaces such that each partition has a nice representation. Recall:

Definition 1.2. A representative vector is any of the non-zero vectors from the 1-dimensional subspace corresponding to a point $[v] \in P(V)$.

Then if $[x] = [\lambda x] = [a]$, x and a are both representatives for the same projective point.

We also want to define the notion of the homogoenous coordinates for each projective point, which are just the real points that exist in the corresponding vector subspace.

Definition 1.3. The homogoenous coordinates for $[v] \in P(V)$ are the set $[(x_0 \cdots x_n)]$ equivalent under scalar multiplication by λ.

If we construct a subset of homogoenous coordinates U_0 where $x_0 \neq 1$, notice that each $[(x_0 \cdots x_n)] = [1 \cdots x_n/x_0]$, so $U_0 \cong R^n - 1$. We are left to "partition" the coordinates where $x_0 = 0$, but this is exactly the set of 1-dimensional subspaces of $V^n - 1$, so $P(R^{n-1})$.

1.2 Applications

1.3 Linear Subspaces

We begin by proving a result from elementary linear algebra.

Theorem 1.4. *Let W_1 and W_2 be vector spaces. Then* $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

Theorem 1.5. *In a projective plane $P(V)$, two projective lines, $P(U)$ and $P(U')$, intersect in a unique point.*

Proof. From elementary linear algebra, $\dim V \geq \dim U + U'$. We have shown that $\dim U + U' = \dim U + \dim U' - \dim U \cap U'$. Then $1 \leq \dim U \cap U' \leq 2$. Because $P(U)$ and $P(U')$ are distinct, $\dim U \cap U' = 1$. So $P(U \cap U')$ is a projective point. $\square$

It is useful to think about this result using our model of the projective plane as a sphere and using our decomposition.

We can think of projective lines as planes in $\mathbb{R}^3$ that intersect the sphere in two great circles. These great circles intersect in a pair of antipodal points, which is a projective line.

Alternatively

1.4 Projective Transformations

Given a linear transformation $T : V \to W$, we might want to recover a bijective map $\tau : P(V) \to P(W)$.

Definition 1.6. If T is invertible, τ is a projective transformation between projective spaces.

This is a map of lines to lines.

It seems natural to define τ as $[x] \mapsto [T(x)]$ for any $x \in P(V)$. But notice that because there is no 0 in $P(W)$ (as the collection of 1-dimensional subspaces), $\dim T([x]) = 1$ if τ is to be well-defined over its codomain. Then T must be invertible.

Note. It is not immediately obvious that projective transformations describe bijections (or isomorphisms between underlying vector spaces). However, a T that takes any U in V to 0 induces an ill-defined τ (no 0 in $P(W)$). Furthermore, if $T(U) = T(U')$ for distinct U, U', T cannot be invertible so there will exist some U'' where $T(U'')$ goes to 0. So the requirement that T be invertible is really a requirment that τ needs to be a well-defined map with only 1-dimensional subspaces in its codomain.

Note. Projective transformations are also called *homographies* and have roots in the non abstract origins of projective geometry as a tool to study perspective. In broad strokes, a homography just describes a transformation of perpsectives of the same underlying object.

In fact, projective transformations descibe a collection of linear transformations T that are equivalent up to scalar multiplication.

Proposition 1.7. *If $T, T' : V \to W$ define the same projective transformation, $T = \lambda T'$.*

Proof. If V is generated by basis $\{v_1 \cdots v_n\}$, then $[Tv_i] = [T'v_i]$ by assumption. Certainly for each basis element, $Tv_i = \lambda_i T'v_i$. For an arbitrary element, $\sum Tv_i = \sum \lambda_i T'v_i$, our assumption tells us $T(\sum v_i) = \lambda T'(\sum v_i)$. Then, by linearity:

$$\lambda T'(\sum v_i) = T(\sum v_i) = \sum Tv_i = \sum \lambda_i T'v_i$$

. So $\lambda = \lambda_i$. Because $\lambda x = \lambda \sum v_i$, $T = \lambda T'$. □

In the real projective plane, there is very a natural geometric picture one can construct to see these transformations are bundles of lines related by a disjoint "observing point".

Example 1.8. Consider two projective lines $P(U)$ and $P(U')$ in the projective plane $P(V)$. If there exists a point $K \in P(V)$ disjoint from both lines, this point induces a natural projective transformation τ. For each point (line) in $P(U)$, draw a line through K, and where it meets $P(U')$ is its image under τ.

We can see this is indeed a projective transformation by proving the underlying linear transformation $T : U \to U'$ is invertible. If W is the subspace corresponding to K, any $a \in U$ can be uniquely expressed as $w + a'$ from $W \bigoplus U'$ (W is disjoint from both U and U'). Then $a' = a - w$ where the w component guarantees that $\ker T = 0$.

Note. As outlined here, visualizing the above example in $\mathbb{R}^2$ leads to a natural image of an observer (our extra point) connecting two planes together using their perspective. In fact, this transformation is also called a perspectivity in computer graphics for this reason.

In vanilla linear algebra, we can fully characterize a linear transformation from an n dimensional space by observing what it does to n linearly independent vectors.

Definition 1.9. Points $X_1 \cdots X_{n+1} \in P(V)$ are in general position if any subset of n points have representative vectors that are linearly independent.

Theorem 1.10. *If $X_1 \cdots X_{n+2} \in P(V)$ (in n dimensional $P(V)$) are in general position in $P(V)$ and $Y_1 \cdots Y_{n+2}$ are in general position in $P(W)$, then there is a unique projective transformation such that $\tau(X_i) = Y_i$.*

Proof. We can choose representatives, $v_i \in V$, such that $n+1$ representatives form a basis of V. We can choose representatives such that $v_{n+2} = \sum_{i=0}^{n+1} v_i$. Note that the sum of vectors is unique by linear indpendence and must exist because $(v_i)_i$ form a basis for V.

Similarly, we can choose representatives from W such that $w_{n+2} = \sum_{i=0}^{n+1} w_i$. Again this sum of elements is unique.

Then there exists a unique and invertible $T : V \to W$ described by the mapping of basis elements (where $T(v_i) = w_i$) that induces a projective transform with the desired properties

To see uniqueness, consider an alternative $T' : V \to W$ such that $T'(v_i) = \mu_i w_i$, taking our basis elements to a different representative of a point in $P(W)$.

Then $T'(v_{n+2}) = \mu_{n+2} w_{n+2} = \sum_{i=0}^{n+2} \mu_i w_i = \sum_{i=0}^{n+2} T'(v_i)$. Because w_{n+2}, is the unique sum of representatives expressed earlier, $\frac{\mu_i}{\mu_{n+2}} = 1$. So $\mu_i = \mu_{n+2}$ and $T = \mu_{n+2} T'$. □

Note. Any two distinct points on the projective line are linearly independent vectors, so any three distinct points on the projective line are in general position.

Theorem 1.11 (Desargues' Theorem)**.** *Consider points $A, A', B, B', C, C' \in P(V)$, where the lines AA', BB', CC' are distinct and concurrent (intersect in a single point). Then the three points of intersection $AB \cap A'B'$, $BC \cap B'C'$, $CA \cap C'A'$ are collinear.*

Proof. Denote the point of intersection between our three lines P. Because any three points on the projective line are in general position, we can express p as three different linear combinations of vectors:

$$p = a + a' = b + b' = c + c'$$

Where a is a representative from A and so forth in the obvious way. We can make new points.

$$c'' = a - b = b' - a'$$
$$a'' = b - c = c' - b'$$
$$b'' = c - a = a' - c'$$

Notice that all of a'', c'', b'' lie in a two-dimensional vector subspace and are representatives of projective points at our desired intersections: $[c''] \in AB \cap A'B'$ and so on.

Furthermore, notice $c'' + a'' + b'' = 0$, and this expression is 0 for any linear combination of these vectors. To see this, consider

$$\lambda_c c'' + \lambda_a a'' + \lambda_b b'' = 0$$

and expand this expression to

$$\lambda_c(a - b) + \lambda_a(b - c) + \lambda_b(c - a) = 0$$

.

$$(\lambda_c - \lambda_b)a + (-\lambda_c + \lambda_a)b + (\lambda_a + \lambda_b)c$$

We can then see we need to choose new representatives from our projective lines that will lead to the desired expression.

$$p = (\lambda_c - \lambda_b)a + a' = (-\lambda_c + \lambda_a)b + b' = (\lambda_a + \lambda_b)c + c'$$

Because a'', b'', c'' are linearly independent and each exist in a two-dimensional subspace of V, their corresponding points in $P(V)$ are in general position. Three projective points in general position define a line, so these points are collinear as desired.

□

1.5 Duality

We first review definitions of duality from linear algebra.

Definition 1.12. For a vector space V defined over field F, the dual of V is the vector space V' with elements that are linear transformations $f : V \to F$.

Definition 1.13. If the basis of V is $\{v_1 \cdots v_n\}$, V' has a corresponding basis where for each v_i, $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for all $j \neq i$.

Given a linear transformation $T : V \to W$ over vector spaces, there is a canonical linear transformation induced over its duals $T' : W' \to V'$, given by $T'(f) = f(T)$. Indeed $T'f = fT : V \to F$ and $T'f = fT = v_i \mapsto f(T(v_i))$.

Note. We can use the language of contravariant functors to illustrate the correspondance between linear transformations in their vector and dual spaces.

The correspondence of a category of functions to another category with the domain and codmain swapped is ubituiquitous in category theory and is used to introduce covariant and contravariant functors in Leinster.

Definition 1.14. The dual of a dual, denoted V'', is a vector space of linear transformations $(V \to F) \to F$. This space is isomorphic to V.

To see this isomorphism, take $S : V \to V''$, $Sv = f \mapsto f(v)$. To see it is linear in v:

$$
\begin{aligned}
&S(\lambda_1 v_1 + \lambda_2 v_2) \\
&= f \mapsto f(\lambda_1 v_1 + \lambda_2 v_2) \\
&= f \mapsto (\lambda_1 f(v_1) + \lambda_2 f(v_2)) \\
&= \lambda_1 (f \mapsto f(v_1)) + \lambda_2 (f \mapsto f(v_2)) \\
&= \lambda_1 S(v_1) + \lambda_2 S(v_2)
\end{aligned}
$$

Because f is itself a linear transformation. The kernel of S is 0 as there exist no non zero vector v where $f(v) = 0 \forall f \in V'$. Then S is a linear transformation over spaces with the same dimensions and a non-zero kernel, so it is an isomoprhism.

Definition 1.15. The annihilator U^{oo} for some subspace $V \subseteq U$ is defined $\{f' \in V' \mid f'(f) = 0, \forall f \in U^o\}$

Then we claim $S(U) = U^{oo}$ for any subspace U. Indeed for any $u \in U$, $Su(f) = f(u)$ for all such $f \in U^o$, so $S(U) \subseteq U^{oo}$. Equality follows because S is an isomorphism. (The notes claim this instead follows from the theorem $\dim V = \dim U + \dim U^o$, but I fail to see this)

This fact will be useful in the next section.

1.6 Projective Dual Spaces

If there exists some dual V' for each V, we want to say something about $P(V')$ in its relation to $P(V)$ and maybe use this structure to prove theorems in the projective space using the dual projective space.

The simplest example of this is "two distinct points define a unique line" has an equivalent dual analogue "two distinct lines define a unique point" (and note that in projective spaces, this statement of course holds for parallel lines).

First, examine a point $f \in P(V')$ and notice $\ker f$ is some $U \subseteq V$ with $\dim U = n - 1$. ($f : V' \to F$ and $\dim \ker f = \dim V' - \dim F$). This point then defines, up to scalar multiplication of f, a unique subspace of $P(V)$, a hyperplane. This gives us a correspondence between the vector and dual space that is not possible in the non projective spaces.

Definition 1.16. A hyperplane in $P(V)$ of dimension n is some $P(U)$ of dimension $n - 1$.

Theorem 1.17. *A subspace $P(W) \subseteq P(V')$ of dimension m (where $P(V')$ is dimension n) corresponds to a set of hyperplanes in $P(V)$ that share some fixed subspace $P(U)$ of dimension $n - m - 1$*

Projective dual spaces give us an intepretation of the space of lines in $\mathbb{R}^2$. Recall the familiar decomposition $P^2(\mathbb{R}) = \mathbb{R}^2 \cup P(\mathbb{R})$, where every projective line that is not the line at infinity intersects $\mathbb{R}^2$ in a straight line. Then we need only to remove the line at infinity to describe the space of straight lines, or a point in the dual space.

We parameterize our sphere with spherical coordinates but restrict the range of θ to exclude the north and south poles.

$$x_1 = \sin\theta\cos\phi, x2 = \sin\theta\sin\phi, x3 = \cos\theta; 0 < \theta < \pi, 0 \leq \phi \leq 2\pi$$

With antipodal map is $(\theta, \phi) \mapsto (\pi - \theta, \pi + \phi)$.

Then to represent the space of lines, we need only to sketch the set of antipodal points. Consider $(\theta, \phi) \in (0, \pi) \times [0, 2\pi]$ (gemoetrically this is a half sphere without poles). We only need to identify the bottom edges of this square as the restriction over ϕ means the only antipodal points are of the form $(\theta, 0)$ and (θ, π).

This identification of the top and bottom edge of the square, in reverse orientation, yields the mobius strip.

Why was it important that we removed the poles? In other words, why do we need an open set $\theta \in (0, \pi)$ as the bottom edge of our square?

Need to return to this.

1.7 The Fundamental Group of the Projective Plane

This was all to become more comfortable with the projective real plane

Computing $\pi(P^2(\mathbb{R}), x_0) \cong \mathbb{Z}_2$ is actually straightforward from van Kampen's applied to 2-dimensional cell complexes, but understanding the result geometrically is more challenging.

Recall the following result from Hatcher:

Theorem 1.18 (van Kampen's applied to 2-dimensional cell complexes)**.** *Consider a 1-skeleton X. If we attach a collection of 2-cells to X with maps φ_α, we obtain a 2-skeleton Y. Let $N =< \gamma_\alpha \varphi_\alpha \bar{\gamma_\alpha} >$ be the loops around the attaching 2-cells (where the γ_α connect the loops to the basepoint in X). Then $\pi(X)/N \cong \pi(Y)$.*

The proof of this theorem gives us a general strategy for computing $\pi(X)$ and we will use this to solve $\pi(P^2(\mathbb{R}))$.

Consider $P^2(\mathbb{R})$ as a CW complex, where the 1-skeleton is S^1 and the 2-skeleton is the open disk D^2 attached to S^1 along the loop aa^{-1}.

Let B be the attached disk and A be the entire projective plane with a hole in it. Then $P^2(\mathbb{R}) = A \cup B$ where both A and B are open, so van Kampen's tells us $\pi_1(P^2(\mathbb{R})) \cong (\pi_1(A) \star \pi_1(B))/N$, where N is generated by the loops of the attachment map.

$\pi_1(B)$ is 0 as the open disk is contractible, while A deformation retracts to the circle so $\pi_1(A) \cong \mathbb{Z}$. What is left is to describe N, but we know from our theorem that $\varphi : S^1 \to S^1$, that describes how to "glue" our disk to our 1-skeleton, is also a loop that generates N, and this is just a^2.

Another way to see that $N =< a^2 >$ is to compute it. $N = \{i_{AB}(\omega) i_{BA}(\omega) \mid \omega \in \pi_1(A \cap B)\}$, where $i_{AB} : \pi_1(A \cap B) \to \pi_1(A)$. But because the image of i_{BA} is trivial, N is generated by just the loops $i_{AB}(\omega)$. These loops are not trivial because they are included in A and must be pushed to the boundary of the space, so they homotope to the loop that follows the edges of the attached disk, and this is a^2.

So our result is the group free in one generator modulo the relation aa^1:

$$\pi_1(P^2(\mathbb{R})) \cong < a \mid a^2 > \cong \mathbb{Z}_2.$$

Now we should think a bit about what this result means.

Note (Geometric interpretation of loops on mobius band)**.** (What the fuck was I trying to say?)

Pick a basepoint on the mobius band and construct a loop by traversing the band until you have arrived back at the basepoint. See this is not nullhomotopic.

Now continue from the same basepoint and trace the same path. See that the combined loop is now nullhomotopic.

1.8 Application

It can be shown that $\pi_1(SO(3)) \cong \mathbb{Z}_2 \cong \pi_1(P^3(\mathbb{R}))$ and the 2-skeleton of $\pi_1(P^3(\mathbb{R})$ agrees with $\pi_1(P^2(\mathbb{R}))$ (using techniques I do not yet understand so will just take at face value for the time being).

But this result means that the group of rotational symmetries of an object is isomorphic to Z_2. In other words, a rotation of an object *with strings attached* by 360 degrees does not yield the same object, but another rotation does.

In physics, this is called the **Feynman plate trick** (or Dirac belt trick), illustrated extending ones arm with a plate on it and rotating the arm in a full rotation. The plate will be in the same orientation upon the first rotation but the arm will be twisted. Twisting the arm again will yield both the plate and arm in the original orientation.

Note. Binasuan Dance There is a traditional Phillipine dance where the performers will hold glasses of rice wine and elegantly twist their limbs without spillling the liquid.

- Feynman's plate trick
- Binasuan Dance
- An arm rotating a glass

1.9 Exercises

Solutions to selected exercises from Hitchin's notes.

Exercise 1.1. Denote the set of 1-dimensional subspaces of $\{\text{span}\{x, y\} | x \in U_1, y \in U_2\}$ as A. Denote the set of 1-dimensional subspaces of $U_1 + U_2$ as B.

To see $A \subseteq B$, for $a \in A$, a one-dimensional subspace of $\text{span}\{x, y\}$ can be represented by $x + y$, and this is clearly a one-dimensional subspace of $U_1 + U_2$.

To see $B \subseteq A$, choose some basis vector from $b \in B$, then $b = x' + y'$, then it is a one-dimensional subspace of some element of A as desired.

Note that A is exactly the set of projective points defined by all lines through $X \in P(U_1)$ and $Y \in P(U_2)$. And B follows from the definition of $P(U_1 + U_2)$. Their equivalence proves our result.

1.10 Appendix

Theorem 1.19. $\dim W_1 + W_2 = \dim W_1 + \dim W_2$

Proof. Let $S = \{u_1 \cdots u_r\}$ be the basis of $W_1 + W_2$. Let $B_1 = \{u_1 \cdots u_r v_1 \cdots v_s\}$ and $B_2 = \{u_1 \cdots u_r w_1 \cdots w_t\}$ be B extended to be the basis of W_1 and W_2 respectively. If we can show B is the basis of $W_1 + W_2$, we have our result, as $\dim B = r + s + t = (r + s) + (r + t) - r = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

First, we show B is linearly independent. Let

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k = 0$$

.

Notice if we move terms so

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = -\sum_k^t c_k w_k$$

,

then the LHS is in W_1 and the RHS is in W_2, so both sides represent the same element in $W_1 + W_2$. Then $\sum_i^r d_i u_i = -\sum_k^t c_k w_k$, where the LHS uses B and the RHS uses B_2. Again moving terms:

$$\sum_i^r d_i u_i + \sum_k^t c_k w_k = 0$$

Where all c_i must be 0 as B_2 is linearly independent. Then

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = 0$$

But the LHS is described by B_1 which is also linearly independent so all a_i, b_j must also be 0. Then B is linearly independent.

Consider any $w_1 + w_2$.

$$w_1 = \sum_i^r a_i u_i + \sum_j^s b_j v_j$$

$$w_2 = \sum_i^r d_i u_i + \sum_k^t c_k w_k$$

Then

$$w_1 + w_2 = \sum_i^r (a_i + d_i) u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k \in \text{span}\, W_1 + W_2$$

□

Note (Spherical, Cyndrical, Cartesian coordinates)**.** Cyndrical coordinates of a point in $\mathbb{R}^3$ are simply polar coordinates with a z-axis. (θ, r, z):

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$x_3 = z$$

Spherical coordinates use the same θ around the x axis but decribe the point using a distance from the origin p and an angle from the z axis ψ. (θ, ψ, p) is

$$\theta = \theta$$

$$r = \sin\psi$$

$$z = \cos\psi$$

Then the cartesian coordinates for a point in spherical coordinates is:

$$x_1 = \sin\psi\cos\theta$$

$$x_2 = \sin\psi\sin\theta$$

$$x_3 = \cos\psi$$

Where $0 \leq \psi \leq \pi$ and $0 \leq \theta \leq 2\pi$

Note (Quotient space (linear algebra))**.** Let V be a vector space over the field F. If N is a subspace of V, V/N is the set of equivalence classes where $x \equiv y$ if $x = y + n$ for some $n \in N$.

Note (Small theorem on isomorphisms)**.**

Theorem 1.20. *$T : V \to W$ where $\dim V = \dim W$. If $kerT = 0$, T is an isomorphism.*

Proof. T is injective. $T(v) = T(v') \implies T(v) - T(v') = 0 \implies T(v - v') = 0 \implies v - v' = 0 \implies v = v'$. T is surjective. $\dim V = \dim\ker T + \dim\operatorname{im} T$. Then $n = 0 + \dim\operatorname{im} T$ so $\operatorname{im} T = W$. □