Topologist's comb

Kenny Workman

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The topologist's comb is a subspace of $\mathbb{R}^2$ that is one of the classic counterexamples in point set topology used to build intuition around notions of connectedness and path connectedness. It is easy to prove that every path connected space is connected, but the converse need not be true and the reasoning is subtle. This example has been useful in understanding this reasoning and has been useful studying pathologies in deformation retractions.

1 Construction

Let K be the set $\{\{1/n\} \times [0,1] \mid n \in \mathbb{Z}^+\}$. Define $C = \{\{0\} \times [0,1] \cup K \cup [0,1] \times \{0\}\}$ and consider C as a subspace of $\mathbb{R}^2$ with the standard topology. The subspace C is called our **comb space**.

We can remove the left most line, leaving only the point $(0,1)$, giving us the **deleted comb space** D. (Formally $D = \{\{(0,1)\} \cup K \cup ([0,1] \times \{0\})\}$

2 The deleted comb space is not path connected

By leaving behind an isolated point, intuition suggests D would not be path connected and indeed this is the case.

Consider any path $f : [0,1] \to D$ where $f(0) = (0,1)$ and $f(1) = (0,0)$. Our strategy is to show $f^{-1}((0,1))$ is clopen which contradicts the existence of this path.

Define some nbhd of $(0,1)$, V, that does not intersect the x-axis. For each $x \in f^{-1}((0,1))$, we then find some nbhd U of x where $U \subset f^{-1}(V)$. Note that U, as an open interval of $[0,1]$, is connected. We claim that $f(U) = (0,1)$. To see this we proceed by contradiction. If $f(U)$ contains a point other than $(0,1)$, some $(1/n, y)$, then $A = \{(x,y) \in f(U) \mid x < 1/n\}$ and $B = \{(x,y) \in f(U) \mid x >= 1/n\}$ form a clear disconnection.

It is important to understand why this is not true if there is no "isolated" point $(0,1)$, such as in the regular comb space C. The image of U about $x \in f^{-1}((0,1))$ need not be disconnected as $f(U) \subset \{0\} \times [0,1]$. (Open subsets of x could be mapped laterally along the connected "spikes" and such a spike always exist despite choice of V).

Because this argument holds for each $x \in f^{-1}((0,1))$, $f^{-1}((0,1))$ is open, contradicting the existence of our path.

3 The deleted comb space is connected

What is less obvious intuitively is that D is connected, despite the proof being straightforward.

Consider $E = \{K \cup [0,1] \times \{0\}\}$ (the comb space without the left-most line). This space is connected. Note that $\overline{E} = C$ and $E \subset D \subset C$. Because D lies in the closure of a connected space, it is itself connected.

4 The deleted comb space cannot be deformation retracted to points on the "spikes"

As before, consider $(0, 1)$, and construct some homotopy h between C and $(0, 1)$. Define an open set V about $(0, 1)$ that does not intersect the x-axis. By construction, $(0, 1) \times I \subset h^{-1}(V)$. The tube lemma gives us some U small enough such that $(0, 1) \times I \subset U \times I \subset h^{-1}(V)$. But this suggests there exists a path from points on $(1/n, 1)$ to $(0, 1)$ for some other n that does not go through the x-axis. This is not possible so no such homotopy can exist.