5 - Limits

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July 7, 2023

Problem 5.1.34.

Solution. We can prove the converse directly. If the following commutative square is a pullback:

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Then the following diagram commutes for any other object A in the category where $gj = fj$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"] \arrow[drr, "j"] \arrow[ddr, "j"'] & & \\
& E \arrow[r, "i"] \arrow[d, "i"] & X \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

.

Note there exists a unique $h : A \to E$ where $ih = j$ and $gi = fi$. This diagram collapses to:

$$\begin{tikzcd}
A \arrow[dr, dashed, "h"'] \arrow[drr, "j"] & & & \\
& E \arrow[r, "i"'] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

Then A is a fork as $gj = fj$ from before. And indeed there exists a unique $h : A \to E$ where $ih = j$. This is true for any such fork in this category, which is a special case of the commutative square describe before, so E is an equalizer.

$\square$

Problem 5.1.38(a).

Solution. We must show that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone of $D : \mathbf{I} \to \mathscr{A}$.

First we show that L is a cone; that for any $u : J \to K$ (where $J, K \in \mathscr{A}$). $Dup_J = p_K$. We know that L is a fork of s and t (in particular it is the equalizer of s and t) so $sp = tp$. Then for any $u : J \to K$, $s_u p = t_u p$. Because $t_u = pr_K$, $t_u p = p_K$. Similarly, because $s_u = D_u pr_J$, $s_u p = D_u p_J$. Then $p_K = D_u p_J$ so L is a cone.

We finish by showing that L is also the limit of D. Notice that any other cone A on D is also a fork of s and t (The family of maps $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ can be represented as the single $f : A \to \prod_{I \in \mathbf{I}} D(I)$ with the property $sf = tf$ equivalent to $Duf_J = f_K$ for any $u : J \to K$). Because L is the equalizer of s and t, there exists a unique $g : A \to L$ such that $pg = f$ for any such cone A. Then certainly $p_I g = f_I$ so L is also a limit cone.

$\square$

Problem 5.1.38(b).

Solution. We first that if $\mathscr{A}$ has binary products and a terminal object, $\mathscr{A}$ also has finite products.

Consider $X, Y, Z \in \mathscr{A}$. Because $\mathscr{A}$ has binary products, both the products $X \times Y$ and $(X \times Y) \times Z$ exist.

Define $X \times Y$ as the product with projections p_X and p_Y and P as the product $(X \times Y) \times Z$, with projections $p_{X \times Y}$ and p_Z. We will show that P also has projections $p_X p_{X \times Y}$, $p_Y p_{X \times Y}$, p_Z onto X, Y, and Z respectively that satisfy the necessary properties so that it is also the product $X \times Y \times Z$.

We pick an arbitrary object and set of maps:

$$\begin{tikzcd}
& X \\
A \arrow[ur, "f_X"] \arrow[r, "f_Y"'] \arrow[dr, "f_Z"'] & Y \\
& Z
\end{tikzcd}$$

And claim that for any such diagram, there exists a unique $g : A \to P$, where $f_X = p_X p_{X \times Y} g$, $f_Y = p_Y p_{X \times Y} g$ and $f_Z = p_Z g$.

To see $f_X = p_X p_{X \times Y} g$, we introduce a terminal object T and construct the following commutative diagram:

$$\begin{tikzcd}
& & & X \arrow[r, "t_X"] & T \\
A \arrow[urrr, bend left, "f_X"] \arrow[r, "g"] & P \arrow[r, "p_{X \times Y}"] \arrow[dr, "p_Z"'] & X \times Y \arrow[ur, "p_X"] \arrow[r, "p_Y"] & Y & \\
& & Z & &
\end{tikzcd}$$

Where g is the map satisfying the unversal property for P as the product $(X \times Y) \times Z$. Then $g p_{X \times Y} p_X t_X = f_X t_X$, because T is a terminal object, so $g p_{X \times Y} p_X = f_X$. An identical argument proves $g p_{X \times Y} p_Y = f_Y$. The result $g p_Z = f_Z$ can be seen considering $(X \times Y) \times Z$ as an ordinary binary product.

We can continue in this way to build products of any finite set of objects in $\mathscr{A}$.

With finite products and equalizers, the argument in **5.1.38(a)** remains the same when D has a finite number of maps $u \in \mathbf{I}$. Then D has finite limits.

(Its actually still unclear to me why we can't make products of infinite objects)

$\square$

Problem 5.2.21.

Solution. We first prove $s = t$ iff there exists an equalizer of s and t the given category and its an isomorphism.

If $s = t$, the equalizer of s and t is X along with the identity 1_X. This equalizer is certainly a fork as $s1_X = t1_X$. For any other fork A with $f : A \to X$ where $sf = tf$, f itself is the unique map such that $s1_X f = t1_X f$. This proves our forward argument.

If there exists an equalizer of s and t that is also an isomorphism, which we shall denote as the object E with map $i : E \to X$, then $si = ti$. If we precompose these maps with the inverse of i denoted j, then $sij = tij$ is the same as $s1_X = t1_X$ and $s = t$, which is our desired result.

We then prove there exists an equalizer of s and t and its an isomorphism iff there exists a coequalizer of s and t and its an isomorphism. To see the forward direction, denote our equalizer as E with map $i : E \to X$ and consider the (co?)fork Y with isomoprhism 1_Y. Certainly $1_Y si = 1_Y ti$ and precomposing with the inverse j of i we recover the desired $1_Y s = 1_Y t$. Y is also a coequalizer as any (co?)fork A with $f : Y \to A$ where $fs = ft$ induces $f1_Y s = f1_Y t$.

The proof the reverse direction follows an identical structure and is left as an exercise to the reader.

$\square$

Problem 5.2.22(a).

Solution. The coequalizer is the set of equivalence classes of X generated by the relation $R = \{(f(x), x) | x \in X\}$ denoted by X^*, along with the map $p : X \to X^*$ which sends each $x \in X$ to its respective equivalence class. Indeed $pf = p1$.

We can verify that this coequalizer is universal in this property. Consider any cofork A, with map $h : X \to A$ with $hf = h1$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "1"'] & X \arrow[r, "p"] \arrow[rr, bend right, "h"'] & X^* \arrow[r, "g"] & A \end{tikzcd}$$

We define our unique g as $g = x^* \mapsto h(x)$ where x is an arbitrary member of the equivalence class x^*. We can see that h and pg are then the same map. $pg = x \mapsto h(x)$ such that $h(x) = h(f(x))$ by our construction of g. This is an alternative way of stating $h = x \mapsto h(x)$ where $hf = h$.

□

Problem 5.2.22(b).

Solution. Similar to (a), the coequalizer in **Top** is the space whose underlying set is the equivalence classes of X generated by the relation $R = \{(f(x), x) | x \in X\}$ denoted by X^*, along with the map $p : X \to X^*$ which sends each $x \in X$ to its respective equivalence class. The space X^* inherits the topology induced by the quotient map $\{U \subset X^* | p^{-1}(U) \text{ open in } X\}$. Because p is strongly continuous, as $U \subset X^*$ open in X^* iff $p^{-1}(U)$ open in X, p is certainly continuous.

We can verify this coequalizer is universal in this property. Consider any cofork, with continuous map $h : X \to A$ with $hf = h1$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "1"'] & X \arrow[r, "p"] \arrow[rr, bend right, "h"'] & X^* \arrow[r, "g"] & A \end{tikzcd}$$

Similar to (a), we define our universal map g as $g = x^* \mapsto h(x)$, where x is any x in the equivalence class x^*. We've already shown that $h(x) = gp(x)$ for any $x \in X$ in (a). Then because h is continuous (open $U \subset A \implies h^{-1}(U)$ open in X as given) so is gp and so is g.

If $X = S^1$, $f = [] \mapsto [0, x]$

□

Problem 5.2.24(a).

Solution. We begin by proving the forward direction. Given isomorphic $e, e' \in \mathbf{Epic}(A)$ we must show they induce the same equivalence relation on A.

Recall a function $h : X \to Y$ induces an equivalence relation on X defined as $\{(x, y) | h(x) = h(y)\}$. Then for two functions $h, h' : X \to Y$, if $(h(x) = h(y) \iff h'(x) = h'(y)$ then e, e' induce the same equivalence relation on A.

Consider the following commutative diagram showing our two objects $e, e' \in \mathbf{Epic}(A)$ with isomorphism f:

$$\begin{tikzcd} A \arrow[d, "e"'] \arrow[dr, "e'"] & \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

Note that $fe = e'$ and $e = f^{-1}e'$, using the fact that these maps commute in our category and f is an isomorphism.

If $e(x) = e(y)$, then $f^{-1}e'(x) = f^{-1}e'(y)$ and $e'(x) = e'(y)$ (as an isomorphism in **Set**, f is a bijection). An identical argument can be used to show the converse. Then $e(x) = e(y) \iff e'(x) = e'(y)$ and we shown this is an alternative way of stating that e, e' induce the same equivalence relation on A.

To see the reverse argument, that given $e, e' \in \textbf{Epic}(A)$ that induce the same equivalence relation on A these functions must be isomorphic, we will construct an bijection $f : X \to X'$ such that $fe = e'$ and $e = f^{-1}e'$.

We define $f = e(a) \mapsto e'(a)$ and claim this mapping is a bijection. To see f is injective, recall if $e, e' \in \textbf{Epic}(A)$ induce the same equivalence relation on A, $e(a) = e(a') \iff e'(a) = e'(a')$. Then $e(a) \neq e(a') \implies e'(a) \neq e'(a')$. To see f is surjective, note that e' is surjective and if $x = e'(a) \in X'$ exists, certainly $e(a)$ exists. To see that f itself is well-defined, note that e is surjective. Since f is a bijection, e, e' are isomorphic.

This proves our result, that each equivalence relation on A corresponds to an isomorphic class of functions out of A.

$\square$

Problem 5.2.24(b).

Solution. Fix some group $G \in \textbf{Grp}$ and construct the full subcategory $\textbf{Epic}(G)$ of $G\backslash Grp$ whose objects are epics.

Our "quotient objects" in this subcategory are the isomorphism classes of epics. We will show that each such isomorphism class corresponds to a unique normal subgroup of G.

Consider two such epics ψ, ψ' with isomorphism ϕ:

$$\begin{tikzcd} G \arrow[d, "\psi"'] \arrow[dr, "\psi'"] & \\ X \arrow[r, "\phi"] & X' \end{tikzcd}$$

We claim $ker(\psi) = ker(\psi')$. For any $x \in G$ where $\phi'(x) = 1$, $\psi\phi(x) = 1$ by commutativity and $\psi^{-1}\psi\phi = \phi(x) = 1$ because ψ is an isomorphism. The same argument holds for the converse, so ψ and ψ' share the same kernel.

Then because $ker(\psi) \trianglelefteq G$ for any homormorphism $\psi : G \to X$, the isomorphism class of such epics corresponds to a unique normal subgroup of G.

We now show the reverse, that each normal subgroup of G corresponds to a unique quotient object. Define $N \trianglelefteq G$, and construct an arbitrary surjective homomorphism $\psi : G \to X$ where $ker(\psi) = N$. We claim that any additional homormophism ψ' that is surjective and shares this kernel is isomorphic to ψ. The isomorphism theorems tell us that $\psi(G) \cong G\backslash N$ and $\psi'(G) \cong G\backslash N$. Then $\psi(G) \cong \psi'(G)$ and this proves our result.

$\square$

Problem 5.2.25(a).

Solution. Let $m : A \to B$ be a split monic. We will show this is also a regular monic. Define any $e : B \to A$ such that $em = fm$. We claim that m is the equalizer of e and f. To see this, consider any other fork C, such that $eh = fh$, in the following diagram:

$$\begin{tikzcd} C \arrow[dr, "g"'] \arrow[drr, "h"] & & & \\ & A \arrow[r, "m"] & B \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] & A \end{tikzcd}$$

By construction, $eh = emg$. Then $eh = 1_A g$, so our universal map is defined as $g = eh$. Indeed to see, $emg = fmg$, observe $emeh = fmeh$ and by substitution $1_A eh = 1_A eh$, giving us our result.

We now show that m is also a monic. For any $x, x' : X \to A$, where $mx = mx'$, we show $x = x'$. Notice that since m is an equalizer for some maps s, t, the object X and map mx must be a fork of these maps ($tmx = smx$). If we construct the diagram of this fork along with its projection g on our equalizer:

$$
\begin{tikzcd}
X \arrow[dr, "g"'] \arrow[drr, "mx"] & & & \\
& A \arrow[r, "m"] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A
\end{tikzcd}
$$

We notice that our universal g for mx is exactly x and because it is unique, $mx = mx' \implies x = x'$. $\square$

Problem 5.3.8.

Solution. We define $F : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ as $F = (X, Y) \mapsto X \times Y$ where $X \times Y$ is the binary product of (X, Y). Because $\mathscr{A}$ has binary products, this assignment is straightforward.

To define assignment of morphisms, consider an additional $(X', Y') \in \mathscr{A} \times \mathscr{A}$ with morphism $(f, g) : (X, Y) \to (X', Y')$ induced by $f : X \to X', g : Y \to Y'$. When we treat $X \times Y$ as any other object satisfying the properties of product $X' \times Y'$, the following diagram emerges:

$$
\begin{tikzcd}
& X\times Y \arrow[dl, "p_x"'] \arrow[d, "\bar{g}"] \arrow[dr, "p_y"] & \\
X \arrow[d, "f"'] & X'\times Y' \arrow[dl] \arrow[dr] & Y \arrow[d, "g"] \\
X' & & Y'
\end{tikzcd}
$$

$F((f, g)) = \bar{g}$, where $\bar{g}$ is universal map associated with $fp_x : X \times Y \to X'$ and $gp_y : X \times Y \to Y'$.

We can verify F satisfies the necessary axioms. For any $(A, A) \in \mathscr{A} \times \mathscr{A}$. $F(1_{(A,A)}) = 1_{A \times A}$. We can examine the commutative diagram to verify that the universal map assigned under our definition is the same as the desired identity morphism:

$$
\begin{tikzcd}
& X\times Y \arrow[d, "1_{X\times Y}"] \arrow[ddl, "p_X"'] \arrow[ddr, "p_Y"] & \\
& X\times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \\
X & & Y
\end{tikzcd}
$$

To see composition, we define an additional $(X'', Y'') \in \mathscr{A} \times \mathscr{A}$ with morphism $(f', g') : (X', Y') \to (X'', Y'')$ induced by $f : X' \to X'', g : Y' \to Y''$. We wish to show that $F((f', g') \circ (f, g)) = F((f', g')) \circ F((f, g))$ $\square$

$$
\begin{tikzcd}
& X\times Y \arrow[dl, "p_X"] \arrow[d, "\bar{g}"] \arrow[dr, "p_Y"'] \arrow[dddl, bend right, "f'fp_X"'] \arrow[dddr, bend left, "g'gp_Y"] & \\
X \arrow[d, "f"'] & X'\times Y' \arrow[dl, "p_{X'}"] \arrow[d, "\bar{\bar{g}}"] \arrow[dr, "p_{Y'}"'] & Y \arrow[d, "g"] \\
X' \arrow[d, "f'"'] & X''\times Y'' \arrow[dl, "p_{X''}"] \arrow[dr, "p_{Y''}"'] & Y' \arrow[d, "g'"] \\
X'' & & Y''
\end{tikzcd}
$$

$F((f', g')) \circ F((f, g)) = \bar{\bar{g}}\bar{g}$. $F((f', g') \circ (f, g))$ is equivalent to the universal map induced by $f'fp_X$ and $g'gp_Y$. But it is clear from our diagram that is the same as $\bar{\bar{g}}\bar{g}$, giving us our result.

Problem 5.3.11(a).

Solution. We will show that the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ creates arbitrary limits.

To see this we define a diagram $D : \mathbf{I} \to \mathbf{Grp}$ where $\mathbf{I}$ is any small category. For any limit cone $(B \xrightarrow{q_I} UDI)_{I\in\mathbf{I}}$ on FD, we must show there exists a corresponding limit cone $(A \xrightarrow{p_I} DI)_{I\in\mathbf{I}}$ on D where $U(A) = B$ and $U(p_I) = q_I$ for each $I \in \mathbf{I}$.

We know that B is the set $\{(x_I)_{I\in\mathbf{I}} \in \prod_{I\in\mathbf{I}} UDI | UDu(x_J) = x_K$ for each $UDu : UDJ \to UDK\}$ and each $(B \xrightarrow{q_I} UDI)$ is the projection map.

Then there is a unique group structure we can impose on A that satisfies the desired properties. We know $p_I : \prod_{I\in\mathbf{I}} DI \to DI$ is defined as $(x_I)_I \mapsto x_I$. Then for each $a, a' \in A$, $p_I(a \circ a') = p_I(a) \circ p_I(a') = x_I \circ x'_I$ for each $I \in \mathbf{I}$. So $a \circ a' = (x_I \circ x'_I)_{I\in\mathbf{I}}$. A similar argument recovers inverses for each element and the identity for our group and we have our result.

□

Problem 5.3.11(b).

Solution. We will show that the forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$ creates arbitrary limits.

We can reuse much of the previous argument but must verify that the group structure we define is abelian. As before, $a, a' \in A$, $p_I(a \circ a') = p_I(a) \circ p_I(a') = x_I \circ x'_I$ for each $I \in \mathbf{I}$. So $a \circ a' = (x_I \circ x'_I)_{I\in\mathbf{I}}$. But each DI is abelian, so $(x'_I \circ x_I)_{I\in\mathbf{I}}$ for each $I \in \mathbf{I}$ so certainly $(x'_I \circ x_I)_{I\in\mathbf{I}}$. Then $a \circ a' = a' \circ a$.

□